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Combination of Kharrat-Toma Transform and Homotopy Perturbation Method to Solve a Strongly Nonlinear Oscillators Equation

[1]Prof. Bachir Nour Kharrat [2] Dr. George Albert Toma

[1] Department of Mathematics, Aleppo University, Aleppo, Syria.
[2] Department of Mathematics, Aleppo University, Aleppo, Syria.

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Corresponding Author:

George Albert Toma

*Email:*george.toma.1987@gmail.com

Abstract— This article introduces a new hybridization between the Kharrat-Toma transform and the homotopy perturbation method for solving a strongly nonlinear oscillator with a cubic and harmonic restoring force equation that arising in the applications of physical sciences. The proposed method is based on applying our new integral transform "Kharrat-Toma Transform" and then using the homotopy perturbation method. The objective of this paper is to illustrate the efficiency of this hybrid method and suggestion modified it. The results showed that the modified method is effectiveness and more accurate.

Keywords: *approximate solution, homotopy perturbation method ,initial value problem, Kharrat-Toma transform, strongly nonlinear oscillator.*

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I. Introduction

Nonlinear phenomena are very common in many applications in physics, engineering, mechanics and applied mathematics. Most of these phenomena are modeled with non-linear differential equations, and in some cases these nonlinear differential equations are difficult to solve using traditional techniques, such as the Runge-Kutta method, so the researchers proposed new modifications and new hybrids to find approximate solutions with small errors. A strongly nonlinear oscillator with a cubic and harmonic restoring force equation represents a system consisting of a mass resting on a spring with cubic and quintic nonlinearity [1].

Hosen et.al presented a modified energy balance method to obtain higher-order approximations to the oscillators with cubic and harmonic restoring force [1]. Az-Zo’bi et.al suggested a new modified version of the Adomian-Rach decomposition method (MDM), which is based on combining a series solution and decomposition method for solving nonlinear differential equations with Adomian polynomials for nonlinearities [2]. Beléndez et.al proposed a modification of the generalized harmonic balance method in which analytical approximate solutions have a rational form to find an analytical approximate solution for conservative nonlinear oscillators [3]. Ganji et.al applied He’s Energy Balance Method to solve strong nonlinear Duffing oscillators with cubic-quintic nonlinear restoring force [4]. Rehman et.al, developed homotopy perturbation double Sumudu transform method (HPDSTM) which is obtained by combining homotopy perturbation method, double Sumudu transform and He’s polynomials to find the solution of linear fractional one and two dimensional dispersive KdV and nonlinear fractional KdV equations [5]. Hosen et.al analysed and solved a complicated strongly nonlinear oscillator with cubic and harmonic restoring force by harmonic balance method (HBM) [6]. El-Naggar et.al employed He’s modified perturbation technique to solve nonlinear Duffing oscillators of fifth order in two cases first, without forced term and second with forced term, which hold for all the values of amplitude of the oscillator [7]. Alam et.al introduced a new approach based on the harmonic balance method to obtain approximate solution of some strongly nonlinear oscillators equations [8]. Adamu et.al give analytical solutions to a nonlinear oscillator with coordinate-dependent mass and Euler–Lagrange equation using the parameterized homotopy perturbation method [9]. Kharrat et al. were interested in modifying and hybridizing the homotopy perturbation method with many integral transforms to solve initial or boundary value problems [10-17], and they recently proposed a new integral transform to solve ordinary differential equations [18].

The rest of this paper is ordered as follows: Section 2 introduces an overview of Kharrat-Toma transform and some properties. Section 3 describes the proposed hybrid method and the methodology for solving the strongly nonlinear oscillator equation. Section 4 presents the modification of our proposed hybrid method. Finally, the conclusion is given in Section 5.

II. KHARRT-TOMA TRANSFORM OVERVIEW

The Kharrat-Toma transform proposed by the Syrian researchers Kharrat and Toma in 2020 to solve initial or boundary value problems represented by ordinary differential equations with initial or boundary conditions.

Definition. The Kharrat-Toma integral transform and inversion is defined by

$$B[f(x)] = G(S) = s^3 \int_0^\infty f(x)\, e^{\frac{-x}{s^2}} dx \quad , \quad x \geq 0$$

$$f(x) = B^{-1}[G(S)] = B^{-1}\left[s^3 \int_0^\infty f(x)\, e^{\frac{-x}{s^2}} dx \right]$$

The B integral transform states that, if $f(x)$ is piecewise continuous on $[0, +\infty)$ and has exponential order. The B^{-1} will be the inverse of the B integral transform.

Where

$$B\left[f^{(n)}(x)\right] = \frac{1}{s^{2n}} G(s) - \sum_{k=0}^{n-1} s^{-2n+2k+5} f^{(k)}(0) \quad ; \; n \geq 1$$

The Kharrat-Toma transform of some functions as follows:

(1) $f(x) = 1 \underset{B^{-1}}{\overset{B}{\rightleftarrows}} G(s) = s^5$

(2) $f(x) = x^n \underset{B^{-1}}{\overset{B}{\rightleftarrows}} G(s) = s^{2n+5}.\, n!$

(3) $f(x) = \sin(kx) \underset{B^{-1}}{\overset{B}{\rightleftarrows}} G(s) = \dfrac{k s^7}{1 + k^2 s^4}$

(4) $f(x) = \cos(kx) \underset{B^{-1}}{\overset{B}{\rightleftarrows}} G(s) = \dfrac{s^5}{1 + k^2 s^4}$

(5) $f(x) = sh(kx) \underset{B^{-1}}{\overset{B}{\rightleftarrows}} G(s) = \dfrac{k s^7}{1 - k^2 s^4}$

(6) $f(x) = \cosh(kx) \underset{B^{-1}}{\overset{B}{\rightleftarrows}} G(s) = \dfrac{s^5}{1 - k^2 s^4}$

III. THE PROPOSED HYBRID METHOD (KTHPM)

The strongly nonlinear oscillator with cubic and harmonic restoring force is modelled mathematically by the following nonlinear differential equation as [1]:

$$u'' + u + a u^3 + b \sin(u) = 0 \quad , \quad u = u(x) \tag{1}$$

Where a and b are constants and the initial conditions are:

$$u(0) = A_0 \quad , \quad u'(0) = 0$$

Where $A_0 = 1$ and $a = b = 1$, $b\sin(u)$ is the driving force and $u(x)$ is the system response [1].

The following steps explain the methodology of the proposed hybrid method (KTHPM):

Step 1: Writing $\sin(u)$ by Maclaurin series, then equation (1) can be written as follows

$$u'' + u + u^3 + u - \frac{u^3}{6} + \frac{u^5}{120} - \frac{u^7}{5040} + \frac{u^9}{362880} + \ldots = 0 \tag{2}$$

Taking the Kharrat-Toma transform on equation (2), yields

$$\frac{1}{s^4} B[u] - s\,u(0) - s^3 u'(0) = -B\left[\begin{array}{l} u + u^3 + u - \frac{u^3}{6} + \frac{u^5}{120} \\ -\frac{u^7}{5040} + \frac{u^9}{362880} + \ldots \end{array} \right]$$

Then we have

$$B[u] = s^5 - s^4 B\left[\begin{array}{l} u + u^3 + u - \frac{u^3}{6} + \frac{u^5}{120} - \frac{u^7}{5040} \\ + \frac{u^9}{362880} + \ldots \end{array} \right] \tag{3}$$

Step 2: The homotopy of equation (3) can be written as follows

$$B[u] = s^5 - p s^4 B\left[\begin{array}{l} u + u^3 + u - \frac{u^3}{6} + \frac{u^5}{120} - \frac{u^7}{5040} \\ + \frac{u^9}{362880} + \ldots \end{array} \right] \tag{4}$$

Where $p \in [0,1]$ is an embedding parameter.

According to the HPM the solution of equation (4) can be written as a power series in p

$$u = \sum_{i=0}^\infty p^i u_i \tag{5}$$

Substituting equation (5) into equation (4), yields

$$B\left[\sum_{i=0}^\infty p^i u_i\right] = s^5 - p s^4 B\left[\begin{array}{l} 2\sum_{i=0}^\infty p^i u_i + \frac{5}{6}\left(\sum_{i=0}^\infty p^i u_i\right)^3 + \frac{1}{120}\left(\sum_{i=0}^\infty p^i u_i\right)^5 \\ -\frac{1}{5040}\left(\sum_{i=0}^\infty p^i u_i\right)^7 + \frac{1}{362880}\left(\sum_{i=0}^\infty p^i u_i\right)^9 + \ldots \end{array} \right] \tag{6}$$

Comparing coefficients of the terms with identical powers of p in equation (6)

$$p^0 : B[u_0] = s^5$$

$$p^1 : B[u_1] = -s^4 B\left[2u_0 + \frac{5}{6}u_0^3 + \frac{1}{120}u_0^5 - \frac{1}{5040}u_0^7 + \frac{1}{362880}u_0^9 \right]$$

$$p^2 : B\left[u_2\right] = -s^4 B\left[2u_1 + \frac{5}{2}u_0^2 u_1 + \frac{1}{24}u_0^4 u_1 - \frac{1}{720}u_0^6 u_1 + \frac{1}{40320}u_0^8 u_1\right]$$
$$\vdots$$

Step 3: taking the inverse Kharrat-Toma transform on the result equations, yields

$$u_0 = 1$$

$$u_1 = -\frac{1031113}{725760}x^2$$

$$u_2 = \frac{37752140269}{70230343680}x^4$$

$$u_3 = -\frac{3248022837269691 23}{1274259355729920000}x^6$$

$$u_4 = \frac{712683162000786395943743}{517891623208146173952 0000}x^8$$

Setting $p = 1$, we have the approximate solution of equation (1)

$$u = \sum_{i=0}^{\infty} u_i$$

IV. MODIFIED OUR PROPOSED HYBRID METHOD (M-KTHPM)

In this section, we suggested to modify our proposed hybrid scheme (KTHPM) by adding a new step as in the following order:

Step 1: Derive the strongly nonlinear oscillator equation (1).

$$u^{(3)} = -u' - 3u^2 u' - u'\cos(u) \tag{7}$$

Step 2: We got a new initial condition and the initial conditions became as follows

$$u''(0) = -2 - \sin(1)$$

Step3: Apply the steps of the our proposed hybrid method (KTHPM) that mentioned in section 3 above, i.e Writing $\cos(u)$ by Maclaurin series, then equation (7) can be written as follows

$$u^{(3)} = -u' - 3u^2 u' - u'\left(1 - \frac{u^2}{2} + \frac{u^4}{24} - \frac{u^6}{720} + \ldots\right) \tag{8}$$

Taking the Kharrat-Toma transform on equation (8), yields

$$\frac{1}{s^6}B\left[u\right] - \frac{1}{s}u(0) - s\,u'(0) - s^3 u''(0) = B\begin{bmatrix} -u' - 3u^2 u' \\ -u'\left(1 - \frac{u^2}{2} + \frac{u^4}{24} - \frac{u^6}{720} + \ldots\right) \end{bmatrix}$$

Then we have

$$B\left[u\right] = s^5 + \left(-2 - \sin(1)\right)s^9 + s^6 B\begin{bmatrix} -u' - 3u^2 u' \\ -u'\left(1 - \frac{u^2}{2} + \frac{u^4}{24} - \frac{u^6}{720} + \ldots\right) \end{bmatrix} \tag{9}$$

Step 2: The homotopy of equation (9) can be written as follows

$$B\left[u\right] = s^5 + \left(-2 - \sin(1)\right)s^9 + p\,s^6 B\begin{bmatrix} -u' - 3u^2 u' \\ -u'\left(1 - \frac{u^2}{2} + \frac{u^4}{24} - \frac{u^6}{720} + \ldots\right) \end{bmatrix} \tag{10}$$

Substituting equation (5) into equation (10) and comparing coefficients of the terms with identical powers of p in the result equation, yields

$$p^0 : B\left[u_0\right] = s^5 + \left(-2 - \sin(1)\right)s^9$$

$$p^1 : B\left[u_1\right] = s^6 B\left[-2u_0' - \frac{5}{2}u_0^2 u_0' - \frac{1}{24}u_0^4 u_0' + \frac{1}{720}u_0^6 u_0'\right]$$

$$p^2 : B\left[u_2\right] = s^6 B\begin{bmatrix} -2u_1' - \frac{5}{2}u_0^2 u_1' - \frac{1}{24}u_0^4 u_1' + \frac{1}{720}u_0^6 u_1' \\ -5u_0 u_1 u_0' - \frac{1}{6}u_0^3 u_0' u_1 + \frac{1}{120}u_0^5 u_0' u_1 \end{bmatrix}$$

Step 3: taking the inverse Kharrat-Toma transform on the result equations, yields

$$u_0 = 1 + \left(-2 - \sin(1)\right)\frac{x^2}{2}$$

$$\begin{aligned} u_1 = &-\frac{1}{2592}x^{10} + \frac{1}{31680}x^{12} + \frac{131}{8064}x^8 + \frac{3269}{8640}x^4 \\ &+ \frac{1}{131040}x^{14} - \frac{\sin^4(1)}{552960}x^{16} + \frac{\sin^3(1)}{25344}x^{12} \\ &+ \frac{\sin^5(1)}{1013760}x^{12} - \frac{\sin^2(1)}{230400}x^{16} + \frac{131\sin(1)}{5376}x^8 \\ &- \frac{619}{7200}x^6 + \frac{\sin^2(1)}{34944}x^{14} - \frac{\sin^6(1)}{11059200}x^{16} \\ &- \frac{\sin^3(1)}{5184}x^{10} + \frac{3269\sin(1)}{17280}x^4 + \frac{\sin^4(1)}{101376}x^{12} \\ &+ \frac{\sin^6(1)}{8386560}x^{14} + \frac{\sin(1)}{12672}x^{12} - \frac{\sin^5(1)}{1843200}x^{16} \\ &- \frac{619\sin^2(1)}{28800}x^6 - \frac{619\sin(1)}{7200}x^6 + \frac{\sin^4(1)}{139776}x^{14} \\ &+ \frac{\sin^3(1)}{52416}x^{14} + \frac{\sin^2(1)}{12672}x^{12} + \frac{131\sin^3(1)}{64512}x^8 \\ &- \frac{\sin^2(1)}{1728}x^{10} + \frac{131\sin^2(1)}{10752}x^8 - \frac{\sin(1)}{345600}x^{16} \\ &- \frac{\sin^3(1)}{276480}x^{16} - \frac{\sin^7(1)}{154828800}x^{16} - \frac{\sin(1)}{1296}x^{10} \\ &+ \frac{\sin^5(1)}{698880}x^{14} - \frac{\sin^4(1)}{41472}x^{10} + \frac{\sin(1)}{43680}x^{14} - \frac{1}{1209600}x^{16} \end{aligned}$$

In the following Table 1, we present the comparison of absolute error for the approximate solution by the proposed hybrid method of equation (1) by taking four terms and five terms of the solution series with the modified method (M-KTHPM) by taking two terms and three terms of the solution series. Where (N) is the number of terms of the solution series.

Table I. The absolute error for the oscillator problem

X	KTHPM N=4	KTHPM N=5	M-KTHPM N=2	M-KTHPM N=3
0	2.48922 e – 07	2.12213 e – 08	1.20421 e – 18	3.09671 e –21
0.1	7.67089 e – 06	4.15501 e – 08	6.20148 e – 05	8.01297 e – 09
0.2	4.78936 e – 04	1.57674 e – 05	1.05489 e – 04	3.62149 e – 06
0.3	5.26250 e – 03	3.90991 e – 04	2.01695 e – 03	9.20014 e – 05
0.4	2.81566 e – 02	3.72928 e – 03	2.25810 e – 02	6.01348 e – 04
0.5	1.01106 e – 01	2.09965 e – 02	5.20149 e – 02	8.01232 e – 04

Table I. shows that the results obtained by modified method (M-KTHPM) is accurate and better compared with the hybrid technique (KTHPM).

V. CONCLUSION

In this study, we have proposed a new hybrid scheme combining the Kharrat-Toma transform and the homotopy perturbation method (KTHPM) for solving the strongly nonlinear oscillator with cubic and harmonic restoring force equation represented by nonlinear second order differential equation with initial conditions. In addition, we presented a modification of our proposed method (KTHPM). The obtained results illustrate that the proposed hybrid method (KTHPM) is a powerful and simple technique for obtaining numerical solution of nonlinear initial value problem but the modified scheme is more accurate (M-KTHPM). The computations presented in this work are performed using the Maple software.

REFERENCES

[1] Md. Alal Hosen, G.M. Ismail, A. Yildirim, M.A.S. Kamal, A Modified Energy Balance Method to Obtain Higher-order Approximations to the Oscillators with Cubic and Harmonic Restoring Force, Journal of Applied and computational mechanics, (2020), 6(2), pp. 320-331.

[2] E. A. Az-Zo'bi, K. Al-Khaled, A. Darweesh, Numeric-Analytic Solutions for Nonlinear Oscillators via the Modified Multi-Stage Decomposition Method, Mathematics Journal MDPI, 2019, 7(550).

[3] Beléndez, E. Gimeno, M.L. Álvarez, M.S. Yebra , D.I. Méndez, Analytical approximate solutions for conservative nonlinear oscillators by modified rational harmonic balance method, International Journal of Computer Mathematics, Taylor & Francis, (2008), 1-15.D. D. Ganji, M. Gorji, S. Soleimani, M. Esmaeilpour, Solution of nonlinear cubic-quintic Duffing oscillators using He's Energy Balance Method, Journal of Zhejiang University SCIENCE A, (2009), 10(9):1263-1268

[4] H. U. Rehman, M. Sh. Saleem, A. Ahmad, Combination Of Homotopy Perturbation Method (HPM) And Double Sumudu Transform To Solve Fractional Kdv Equations, Open J. Math. Sci., Vol. 2(2018),No.1, pp. 29 – 38.

[5] Md. Alal Hosen, M.S.H. Chowdhury, A new reliable analytical solution for strongly nonlinear oscillator with cubic and harmonic restoring force, Results in Physics, Elsevier, (2015) 5, 111–114.

[6] A.M. El-Naggar, G.M. Ismail, Analytical solution of strongly nonlinear Duffing oscillators, Alexandria Engineering Journal, Elsevier, (2016) 55, 1581–1585.

[7] M. S. Alam, Md. A. Razzak, Md. Alal Hosen, Md. R. Parvez, The rapidly convergent solutions of strongly nonlinear oscillators, SpringerPlus (2016) 5:1258.

[8] MY. Adamu, P. Ogenyi, AG. Tahir, Analytical solutions of nonlinear oscillator with coordinate-dependent mass and Euler–Lagrange equation using the parameterized homotopy perturbation method, Journal of Low Frequency Noise, Vibration and Active Control, SAGE, (2019), Vol. 38(3–4) 1028–1035.

[9] B. N. Kharrat and G. Toma, Modified Homotopy Perturbation Method by Using Sumudu Transform for Solving Initial Value Problems Represented By System of Nonlinear Partial Differential Equations. World Applied Sciences Journal, (2018), 36 (7): 844-849.

[10] B. N. Kharrat and G. Toma, Differential Transform method for solving initial value problems represented by strongly nonlinear ordinary differential equations. Middle-East Journal of Scientific Research. (2019), 27 (7): 576-579.

[11] B. N. Kharrat and G. Toma, Differential Transform method for solving initial and boundary value problems represented by linear and nonlinear ordinary differential equations of 14th order. World Applied Sciences Journal, (2019), 37 (6), p: 481-485.

[12] B. N. Kharrat and G. Toma, Differential Transform Method for Solving Boundary Value Problems Represented by System of Ordinary Differential Equations of 4 Order. World Applied Sciences Journal. (2020), 38 (1): 25-31.

[13] B. N. Kharrat and G. Toma, Development of Homotopy Perturbation Method for Solving Nonlinear Algebraic Equations. International Journal of Scientific Research in Mathematical and Statistical Sciences. (2020), V 7, Issue 2, pp. 47-50.

[14] B. N. Kharrat and G. Toma, A Hybrid Semi-Analytical Method with Natural Transform for Solving Integro-Differential Equations. International Journal of Scientific Research in Mathematical and Statistical Sciences. (2020), V 7, Issue 2, pp. 47-50.

[15] B. N. Kharrat and G. Toma, A New Hybrid Sumudu Transform With Homotopy Perturbation Method For Solving Boundary Value Problems. Middle-East Journal of Scientific Research. (2020), 28 (2): 142-149.

[16] B. N. Kharrat and G. Toma, Numerical Solution of Van Der Pol Oscillator Problem Using a New Hybrid Method. World Applied Sciences Journal. (2020), 38 (4): 360-364.

[17] B. N. Kharrat and G. Toma, A New Integral Transform: Kharrat-Toma transform and its properties. World Applied Sciences Journal. (2020), 38 (5): 436-443.